ASSIGNMENT SET - I

Department of Mathematics

Mugberia Gangadhar Mahavidyalaya

B.Sc Hon.(CBCS)

Mathematics: Semester-VI

Paper Code: C13T

[Metric Spaces and Complex Analysis]

Complex Analysis

1. Answer all questions

a) Show that $\lim\limits_{z \to 2i} \frac{z^2+4}{z-2i} = 4i$

b) If $f(z) = \begin{cases} \frac{x^3 y(y-ix)}{x^6+y^2}, & z = 0 \\ 0, & z = 0 \end{cases}$

Prove that $\frac{f(z)-f(0)}{z-0} \to 0$ along any radius vector but not as $z \to 0$ in any manner.

c) Show that $f(z) = |z|^2$ continuous for all $z \in \mathbb{C}$.

d) Let $f(z) = |z|^2$. Show that the derivable of f(z) exist only at the origin.

e) Let $f(z) = \begin{cases} \frac{|z|}{Re(z)} \ if \ Re(z) \neq 0 \\ 0 \ if \ Re(z) = 0 \end{cases}$, show that f is not continuous at $z = 0$

2. Answer all questions

a) Define analytic function. State and prove the necessary condition for f(z) to be analytic. Does the condition is sufficient? Give an example.

b) Prove that the function *f(z)* = $\begin{cases} \frac{x^3(1+i)-y^3(1-i)}{x^2+y^2}, z \neq 0 \\ 0, z = 0 \end{cases}$ satisfy Cauchy-Riemann equation at z= 0 but f'(z) does not exist there.

c) Examine the nature of the function

f(z) = $\begin{cases} \frac{x^2y^5(x+iy)}{x^4+y^{10}} \; for \; z \neq 0 \\ 0 \; for \; z = 0 \end{cases}$ at the origin.

d) State and prove the sufficient condition for *w=f(z)=u(x,y)* + *iv(x,y)* to be analytic.

e) Derive the polar form of CR equation.

f) Show that an analytic function with constant modulus is constant.

g) If f(z) is an analytic function of z, then prove that

$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)$ *[Re f(z)]*^{2} = 2 $|f'(z)|^2$.

h) Define Cauchy Riemann equation, Harmonic function.

i) Prove that $\nabla^2 \equiv 4\frac{\partial^2}{\partial z \partial \bar{z}}$

j) If *f(z)* = *u(x,y)* + *iv(x,y)* is an analytic function then *u* and *v* are both harmonic function.

k) Show that the function *u* = $\cos x \cosh y$ is harmonic and find its harmonic conjugate function *v*.

l) Prove that u=$e^{-x}(x\sin y - y\cos y)$ is harmonic. Find v such that *f(z)*= *u*+ *iv* is analytic.

m) Prove that the function u=$e^{x}(x\cos y - y\sin y)$ satisfies Laplace's equation and find the corresponding analytic function *f(z)=u+iv.*

n) If $= x^3 - 3xy^2$, show that there exists a function *v(x,y)* such that *f(z)=u+iv* is analytic in a finite region.

3. Answer all questions

a) State and prove Cauchy's integral formulae.

b) Evaluate on C: $\oint \frac{e^{2z}}{(z+1)4} dz$, where C is the circle |z|=3.

c) State Laurent's theorem and hence define the removable singularity, pole and isolated essential singularity.

d) Find the residue of function f(z) = $e^{-e^{1/z}}$ at z= 0.

e) Evaluate $\int_{|z+1|=2} \frac{z^2}{4-z^2} dz$

f) Find the value of $\oint_{|1-z|=1} \frac{e^z}{z^2-1} dz$.

g) Let C be the counter clockwise oriented circle of radius $\frac{1}{2}$ centered at i . Then find the value of contour integral $\oint \frac{dz}{z^4-1}$ on C.

*h)*Let γ be the positively oriented circle in the complex plane given by $\{z \in \mathbb{C}: |z-1| = 1\}$. Then find the value of $\frac{1}{2\pi i}\int_{\gamma}\frac{dz}{z^3-1}$.

i) Let γ be the positively oriented circle $\{z \in \mathbb{C}: |z| = 3/2\}$. Suppose that $\int_{\gamma}\frac{e^{i\pi z}}{(z-1)(z-2i)^2}dz = 2\pi i C$. Then find the value of $|C|$.

4. Answer all questions (MCQ)

a) The power series $\sum_0^\infty 2^{-n}z^{2n}$ converges if

(A) $|z| \le 2$ (B) $|z| < 2$ (C) $|z| \le \sqrt{2}$ (D) $|z| < \sqrt{2}$

b) The power series $\sum_0^\infty 3^{-n}(z-1)^{2n}$ converges if

(A) $|z| \le 3$ (B) $|z-1| < \sqrt{3}$ (C) $|z-1| \le \sqrt{3}$ (D) $|z| < \sqrt{3}$

c) Consider the power series $\sum_1^\infty z^{n!}$. The radius of convergence of the series is

(A) 0 (B) ∞ (C) 1 (D) a real number greater than 1.

d) Consider the power series f(x) = $\sum_{n=2}^{\infty}\log(n)\,x^n$. Find the radius of convergence.

e) Let γ be the positively oriented circle in the complex plane given by,

$\{z \in \mathbb{C}: |z-1| = \frac{1}{2}\}$, the line integral $\int_{\gamma}\frac{z^{e^{1/z}}dz}{z^2-1}$ equals _____. (A) $i\pi e$ (B) πe (C) $-i\pi$

Metric Space

1. Answer all questions

a) Show that every Cauchy sequence in a metric space is bounded, but the converse is not true.
b) Prove that a convergent sequence $\{x_n\}$ in (X, d) is a Cauchy sequence. Give an example to show that a Cauchy sequence need not be convergent in an arbitrary metric space.
c) Define Cauchy sequence in a metric space. Prove that every convergent sequence is a Cauchy sequence.
d) For any two points x, y, a, b in a metric space (X, d), show that
$$|d(x, y) - d(a, b)| \le d(x, a) + d(y, b).$$
e) Prove that the discrete space (X, d) is a complete metric space.
f) Let (X, d) be a metric space, Prove that the union of an arbitrary collection of open sets of X is open.
g) Prove that a finite set has no limit point.
h) Let (X, d) be any metric space, show that the function $d_1(x, y) = \frac{d(x,y)}{1+d(x,y)}$, $\forall$ x,y $\in$ X is a metric on X.
i) Show that every open sphere is an open set but not conversely.

j) Prove that any closed subset of a complete metric space is complete.
k) Prove that limit of a sequence in a metric space, if exists, is unique.
l) Prove that any closed subset of a complete metric space is complete.
m) When is a metric said to be complete? Give an example of an incomplete metric space.

___________________END_________________